LINEAR-PHASE FIR DIGITAL FILTER DESIGN BY FREQUENCY-SAMPLING METHODS. EXAMPLES

Exercises 4-a.

1. Summary of Important Expressions

Table 1. The four cases of linear phase FIR filters. The real-valued frequency responses. Summary.

M : ***even /odd***	***Symmetry of impulse response***	$H_r(\omega) \in R$
Even	Symmetrical	$H_r(\omega) = 2\sum_{k=0}^{\frac{M}{2}-1} h(k)\cos\omega\left(\frac{M-1}{2}-k\right)$
Odd	Symmetrical	$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{k=0}^{\frac{M-3}{2}} h(k)\cos\omega\left(\frac{M-1}{2}-k\right)$
Even	Antisymmetrical	$H_r(\omega) = 2\sum_{k=0}^{\frac{M}{2}-1} h(k)\sin\omega\left(\frac{M-1}{2}-k\right)$
Odd	Antisymmetrical	$H_r(\omega) = 2\sum_{k=0}^{\frac{M-3}{2}} h(k)\sin\omega\left(\frac{M-1}{2}-k\right)$

Table 2. Summary on the Uniform Frequency-Sampling Method 3.

Unit Sample Response: Symmetric $\alpha = 0$

$$H(k) = G(k)e^{j\pi k/M}, \; k = 0,1,\ldots,M-1$$

$$G(k) = (-1)^k H_r\left(\frac{2\pi k}{M}\right)$$

$$G(k) = -G(M-k)$$

$$h(n) = \frac{1}{M}\left\{G(0) + 2\sum_{k=1}^{U} G(k)\cos\frac{\pi k}{M}(2n+1)\right\}$$

$$U = \frac{M-1}{2} \;\; for\; M\; odd \qquad U = \frac{M}{2}-1 \;\; for\; M\; even$$

Unit Sample Response: Symmetric $\alpha = \frac{1}{2}$

$$H\left(k+\frac{1}{2}\right) = G\left(k+\frac{1}{2}\right)e^{-j\pi/2}e^{j\pi(2k+1)/2M}$$

$$G\left(k+\frac{1}{2}\right) = (-1)^k H_r\left[\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\right]$$

$$G\left(k+\frac{1}{2}\right) = G\left(M-k-\frac{1}{2}\right)$$

$$h(n) = \frac{2}{M}\sum_{k=0}^{U} G\left(k+\frac{1}{2}\right)\sin\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)$$

Unit Sample Response: Antisymmetric $\alpha = 0$

$$H(k) = G(k)e^{j\pi/2}e^{j\pi k/M}, \; k = 0,1,\ldots,M-1$$

$$G(k) = (-1)^k H_r\left(\frac{2\pi k}{M}\right)$$

$$G(k) = G(M-k)$$

$$h(n) = -\frac{2}{M}\sum_{k=1}^{(M-1)/2} G(k)\sin\frac{\pi k}{M}(2n+1) \quad for \;\; M \; odd$$

$$h(n) = \frac{1}{M}\left\{(-1)^{n+1}G(M/2) - 2\sum_{k=1}^{(M/2)-1} G(k)\sin\frac{\pi k}{M}(2n+1)\right\} \quad for \;\; M \; even$$

Unit Sample Response: Antisymmetric $\alpha = \frac{1}{2}$

$$H\left(k+\frac{1}{2}\right) = G\left(k+\frac{1}{2}\right)e^{j\pi(2k+1)/2M}$$

$$G\left(k+\frac{1}{2}\right) = (-1)^k H_r\left[\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\right]$$

$$G\left(k+\frac{1}{2}\right) = -G\left(M-k-\frac{1}{2}\right); \;\; G\left(\frac{M}{2}\right) = 0 \;\; for \;\; odd$$

$$h(n) = \frac{2}{M}\sum_{k=0}^{V} G\left(k+\frac{1}{2}\right)\cos\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)$$

$$V = \frac{M-1}{2} \;\; for \; M \;\; odd \qquad V = \frac{M}{2}-1 \;\; for \; M \;\; even$$

Table 3. Summary on the Uniform Frequency-Sampling Method 4. Recursive FIR Filter Design

$$H(z) = H_1(z)H_2(z)$$

$$H_1(z) = \frac{1 - z^{-M}}{M}$$

$$H_2(z) = \frac{H(0)}{1 - z^{-1}} + \sum_{k=1}^{\frac{M-1}{2}} \frac{A(k) - B(k)z^{-1}}{1 - 2\cos(2\pi k / M)z^{-1} + z^{-2}} \quad \textit{for M odd}$$

$$H_2(z) = \frac{H(0)}{1 - z^{-1}} + \frac{H(M/2)}{1 + z^{-1}} + \sum_{k=1}^{\frac{M}{2}-1} \frac{A(k) - B(k)z^{-1}}{1 - 2\cos(2\pi k / M)z^{-1} + z^{-2}} \quad \textit{for M even}$$

$$A(k) = H(k) + H(M-k) = H(k) + \overline{H(k)} = 2\,\mathrm{Re}\left[H(h)\right] \in R$$

$$B(k) = H(k)e^{-j2\pi k/M} + H(M-k)e^{j2\pi k/M} = 2\left|H(k)\right|\cos\left(\phi_k - 2\pi k / M\right) \in R$$

2. Examples

Example 1
Determine the unit sample response *h(n)* and the frequency response of a linear phase FIR filter of length *M=4* for which the frequency response at $\omega = 0$ and $\omega = \pi/2$ is specified as

$$H_r(0) = 1 \quad H_r\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

For the design, the non-uniform frequency-sampling method has to be applied.

Example 2
Determine the coefficients *h(n)* of a linear phase FIR filter of length *M=16* frequency response of which satisfies the condition:

$$H_r\left(\frac{2\pi k}{16}\right) = \begin{cases} 1 & k = 0,1, \\ 0 & k = 2,3,4,5,6,7,8 \end{cases}$$

For the design, the uniform frequency-sampling method known as non-recursive FIR filter design by direct computation of unit sample response has to be applied.

Example 3
Determine the transfer function *H(z)* of a linear phase FIR filter of length *M=16* frequency response of which satisfies the condition:

$$H_r\left(\frac{2\pi k}{16}\right) = \begin{cases} 1 & k = 0,1, \\ 0 & k = 2,3,4,5,6,7,8 \end{cases}$$

For the design, the uniform frequency-sampling method based on recursive FIR filter design has to be applied.

LINEAR-PHASE FIR DIGITAL FILTER DESIGN BY WINDOWS METHOD. EXAMPLES

Exercise 4-b.

Summary of Important Expressions

Table 1. FIR Linear Time - Invariant System Description: A Review of Basic Expressions

1.	Time – domain description	$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$
2.	Frequency – domain description	$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} = \sum_{k=0}^{M-1} h(k)e^{-j\omega k}$
3.	Impulse response	$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega$

Table 2. Some Commonly Used Windows for FIR Filter Design

	Window Type	**Window Functions,** $w(n)$, $-M \le n \le M$, $M = \frac{N-1}{2}$, $\lvert w(n)\rvert = 0 \;\; for \;\; n > M$
1.	Rectangular	$w(n) = 1$
2.	Bartlett	$w(n) = 1 - \frac{\lvert n\rvert}{M+1}$
3.	Hann	$w(n) = \frac{1}{2}\left[1 + \cos\frac{2\pi n}{2M+1}\right]$
4.	Hamming	$w(n) = 0.54 + 0.46\cos\frac{2\pi n}{2M+1}$
5.	Blackmann	$w(n) = 0.42 + 0.5\cos\frac{2\pi n}{2M+1} + 0.08\cos\frac{4\pi n}{2M+1}$
6.	Kaiser (adjustable window) parameter: α	$w(n) = \frac{I_0\left(\alpha\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\alpha)}$ $\quad I_0(x) = 1 + \sum_{r=1}^{\infty}\left(\frac{(x/2)^r}{r!}\right)^2$

Comments on Kaiser Window: $I_0(x)$ is the modified zero-th-order Bessel function of the first kind. For most practical applications, about 20 terms in the above summation are sufficient to arrive at reasonably accurate values of $w(n)$.

Table 3. Frequency Responses of Some Linear Time-Invariant Systems

	System	Frequency Response
1.	Differentiator	$H\left(e^{j\omega}\right)=\dfrac{j\omega}{T}, \quad -\pi\le\omega\le\pi$.
2.	Hilbert Transformer	$H(j\omega)=\begin{cases}-j & \omega>0\\ 0 & \omega=0\\ j & \omega<0\end{cases}, \quad -\pi\le\omega\le\pi$

Example 1.
Design a band-pass filter with pass-band cut off frequencies $f_1=20\,kHz$ and $f_2=40\,kHz$ of the order $N=11$. Frequency sampling is $f_S=160\,kHz$. It is desired to apply rectangular and Bartlett window at the design.

Example 2.
By the impulse response truncation method (by the windowing method at rectangular window application) design a Hilbert transformer of the order $N=11$.

Example 3.
By the windowing method at Hann window application design a differentiator of the order $N=11$.

Example 4.
Design a stop-band filter with pass-band cut off frequencies $f_1=20\,kHz$ and $f_2=40\,kHz$ of the order $N=11$. Frequency sampling is $f_S=160\,kHz$. It is desired to apply rectangular and Bartlett window at the design.